Nonuniformly elliptic PDEs and variational problems

Trento, September 2-6, 2024

Plenary Talks

Angela Alberico (Napoli)

Fractional Orlicz-Sobolev functions: Embeddings and Continuity properties

The optimal target space is exhibited for embeddings of fractional-order Orlicz-Sobolev spaces. Both the subcritical and the supercritical regimes are considered. In the former case, the smallest possible Orlicz target space is detected. In the latter, the relevant Orlicz-Sobolev spaces are shown to be embedded into the space of bounded continuous functions in $\mathbb{R}^n$. Moreover, their optimal modulus of continuity is exhibited. These results are the subject of a series of joint papers with Andrea Cianchi, Luboš Pick and Lenka Slavíková.

Anna Kh. Balci (Prague/Bielefeld)

Hodge Decomposition in Variable Exponent Spaces with Applications to Regularity Theory

In this talk, we explore the Hodge Laplacian in variable exponent spaces with differential forms on smooth manifolds. We present several results, including the Hodge decomposition in variable exponent spaces and a priori estimates. As an application, we derive Calderón-Zygmund estimates for variable exponent problems involving differential forms and discuss numerical approximations for nonlinear models with differential forms, which have applications in superconductivity. This presentation is based on several works with Swarnendu Sil, Michail Surnachev, and Alex Kaltenbach.

Paolo Baroni (Parma)

Double phase functionals with mild phase transition

We will review some well-established, but also propose some new results for double phase functionals where the energy density is characterized by a phase transition of logarithmic size.

Verena Bögelein (Salzburg)

Higher integrability for sub-critical porous medium type systems

We report on recent results concerning the local higher integrability for the spatial gradient of weak solutions to sub-critical porous medium systems of the type

$$\partial_t u - \operatorname{div}\left(|Du|^{m-1}Du\right) = \operatorname{div} F, \qquad 0 < m < \frac{(N-2)_+}{N+2}.$$

The proof is based on intrinsic scaling and sup-estimates that can be achieved under a certain integrability assumption on the solution. The results are obtained in joint work with Frank Duzaar (Salzburg), Naian Liao (Salzburg) and Ugo Gianazza (Pavia)

Lorenzo Brasco (Ferrara)

Inradius VS. Poincaré

We discuss some estimates on the sharp L^p Poincaré constant of an open set, in terms of various notions of inradius. We start by reviewing some classical two-dimensional results by J. Hersch, E. Makai and W. K. Hayman, together with their extensions. We then present a more general definition of inradius and show that this can be used to completely characterize the validity of the L^p Poincaré inequality. Such a characterization comes with a two-sided estimate of the sharp constant. Some of the results presented have been obtained in collaboration with Francesca Bianchi and Francesco Bozzola.

Dominic Breit (Clausthal)

Strong-type inequalities for Riesz potentials under cocancelling differential constraints

We consider the classical Riesz potential in n dimensions for vectorial functions. We prove estimates in the framework of rearrangement-invariant function spaces under differential constraints. This inlcudes a wealth of function spaces, for instance of Orlicz- and Lorentz-type. Reducing the case of a general co-cancelling differential operator to the plain divergence, a main tool in our approach is a representation for the K-functional between solenoidal Lebesgue spaces. Our result follows then by an interpolation argument coupled with recent optimal endpoint L^1-estimates. This is based on joint work with A. Cianchi and D. Spector.

Iwona Chlebicka (Warsaw)

Refined asymptotics for the Cauchy problem for the fast p-Laplace evolution equation

Our focus is on the fast diffusion equation involving p-Laplacian with $p < 2$ in the whole Euclidean space of dimension $N > 1$. The properties of the solutions to the p-Laplace Cauchy problem change in several special values of the parameter p. In the range of p when mass is conserved, non-negative and integrable solutions behave like the Barenblatt (or fundamental) solutions for large times. By making use of the entropy method, we establish the polynomial rates of the convergence in the uniform relative error for a natural class of initial data. The convergence was established in the literature for p close to 2, but no rates were available. In particular, we allow for the values of p, for which the entropy is not displacement convex, as we do not apply the optimal transportation tools. We approach the issue of long-term asymptotics of the gradients of solutions. In fact, in the case of the radial initial datum, we provide also polynomial rates of the uniform convergence in the relative error of radial derivatives of solutions for $2N/(N+2) < p < 2$. Finally, providing an analysis of needed properties of solutions for the entropy method to work, we open the question on the full description of the basin of attraction of the Bareblatt solutions for p close to 1. This is joint project with Matteo Bonforte and Nikita Simonov.

Giulio Ciraolo (Milano)

Second-order estimates in anisotropic elliptic problems

We discuss Sobolev regularity estimates for the stress field associated to nonlinear second-order elliptic equations with square integrable right-hand side. Particular emphasys is given to problems which arise as the Euler-Lagrange equations of functionals depending on anisotropic norms, and both local and global estimates are obtained. Global estimates are given in domains enjoying minimal assumptions on the boundary. A key point in our approach is a suitable extension of Reilly's identity to the anisotropic and nonlinear settings. This talk is based on diverse joint investigations with C.A. Antonini, A. Cianchi, A. Farina and V. Maz'ya.

Frank Duzaar (Salzburg)

Regularity for (s,p)-harmonic functions

We report on higher Sobolev and Hölder regularity results for local weak solutions of the fractional p-Laplace equation of order $s \in (0,1)$ with $1 < p < \infty$. The relevant estimates are stable when the fractional order s reaches 1, and the known Sobolev regularity estimates for weak solutions of the local p-Laplace equation are recovered. The talk is based on joint work with Verena Bogelein (Salzburg), Naian Liao (Salzburg), Giovanni Molica Bisci (Urbino), and Raffaella Servadei (Urbino).

Franz Gmeineder (Konstanz)

Higher gradient integrability for BD-minimizers - final take

In this talk, I give an update on recent progress on the gradient integrability of bounded BD-minimizers. When considering convex linear growth functionals depending on the symmetric gradients, the usual coercivity space is BD, the functions of bounded deformation. By a basic obstruction, colloquially termed Ornstein's non-inequality, this space is strictly larger than BV, the functions of bounded variation. Yet, as will be discussed in this talk, suitably elliptic variational problems of linear growth then come with essentially the same Sobolev regularity as in the full gradient BV-case. Based on joint work with Lisa Beck & Ferdinand Eitler (Augsburg).

Jonas Hirsch (Leipzig)

Interior regularity for two-dimensional stationary Q-valued maps

The mathematical study of minimal surfaces, i.e. critical points of the area functional, has a long and prolific history. One of the main questions is the dimension of their singular set. The only available result in the general case is Allard's groundbreaking work, On the first variation of a varifold, where he proved that the singular set of a stationary integral varifold is meager. This is based on an regularity result for a stationary varifold close to a multiplicity one plane. Sincthen little to no progress has been made on the question of the optimal dimension of the singular set for integral stationary varifolds. Our article addresses the next interesting situation of multiplicity two, under the assumption that the stationary varifold is a 2-valued Lipschitz graph. Under these assumptions, we are able to confirm the optimal bound that the singular set is indeed of codimension one. In my talk, I would like to present our novelties in particular highlighting two of them. First, we derive an a-priori higher integrability result of the excess measure in the stationary setting. In this step the assumption Q = 2 and the graphicality are essential. But the higher integrability is crucial to be able to "pass" to a linearization. Second, we introduce a new class of Q-valued "gradient Young measures". These integer rectifiable currents allow us to linearize the problem and construct Dirichlet stationary solutions. Despite their measure valued structure, we are able to establish a unique continuation result for them for every Q. This is joint work with L. Spolaor.

Alessandro Iacopetti (Torino)

Shape Optimization and Overdetermined Problems in Cones and Cylinders

In this talk, we present some recent results concerning partially overdetermined problems in unbounded regions. In particular, we focus on the cases of cones and cylinders, investigating the stability and instability of certain classes of solutions that are naturally connected to the geometry of the container. Moreover, we discuss the existence of minimizers of the torsional energy under a volume constraint and their geometric and topological properties. These results are collected in a series of joint works with Prof. F. Pacella (Univ. of Rome "La Sapienza"), Prof. T. Weth (Univ. of Frankfurt), Dott. D. Gregorin (Univ. of Urbino), and Prof. P. Caldiroli (Univ. of Turin).

Lukas Koch (Leipzig)

Uniform regularity results for homogenisation problems of p-Laplace type

I will discuss homogenisation problems of p-Laplace type. It is unclear whether the standard p-growth ellipticity and growth assumptions are stable under homogenisation. I will present regularity results under a set of p-type growth and ellipticity assumptions that are stable under homogenisation. This regularity theory will be used to derive Calderon-Zygmund and large-scale Lipschitz estimates uniform in the homogenisation parameter. The talk is based on joint work with Mathias Schäffner (Halle).

Paolo Marcellini (Firenze)

The Leray-Lions existence theorem under general growth conditions

We discuss about the celebrated existence theorem published in 1965 by Jean Leray and Jacques-Louis Lions. As well known, it is an existence result of weak solutions to a class of Dirichlet problems for second order nonlinear elliptic equations under the so-called natural growth conditions. We describe the possibility to adopt assumptions which allow to handle a class of nonuniformly elliptic PDEs satisfying general growth conditions, such as for instance p-q growth, nowadays largely studied in the literature. In particular we describe some existence and regularity results recently obtained in collaboration with G.Cupini and E.Mascolo, to appear in the Journal of Differential Equations.

Anna Mercaldo (Napoli)

Existence and uniqueness results for elliptic equations with general growth in the gradient

Existence and uniqueness results are established for solutions to homogeneous Dirichlet problems concerning second-order elliptic equations, in divergence form, with principal part a Leray-Lions type operator and a first order term which grows as a q-power of the gradient. The case of elliptic operators having a zero-order term is also considered. Under suitable summability assumptions and smallness on the datum and on the coefficients of the elliptic operators, existence and uniqueness results are presented depending on several ranges of value of the power q of the gradient term. The talk is based on joint papers with A.Alvino and V.Ferone.

Giuseppe Mingione (Parma)
Partial regularity in nonlocal problems

The theory of partial regular regularity for elliptic systems replaces the classical De Giorgi-Nash-Moser one for scalar equations asserting that solutions are regular outside a negligible closed subset called the singular set. Eventually, Hausdorff dimension estimates on such a set can be given. The singular set is in general non-empty. The theory is classical, started by Giusti & Miranda and Morrey, in turn relying on De Giorgi's seminal ideas for minimal surfaces. I will shall present a few results aimed at extending the classical, local partial regularity theory to nonlinear integrodifferential systems and to provide a few basic, general tools in order to prove so called ε-regularity theorems in general non-local settings. From recent, joint work with Cristiana De Filippis (Parma) and Simon Nowak (Bielefeld).

Sunra Mosconi (Catania)
Concavity properties for solutions of the Logarithmic Schrödinger equation

I will discuss the existence of a log-concave solution to the logarithmic Schrödinger equation in a bounded convex domain $\Omega \subseteq \mathbb{R}^N$:

$$\begin{cases} -\Delta u = u \log u^2 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Since the reaction $f(u) = u \log u^2$ is sign-changing and non-monotone, the classical techniques of [1, 2] to attack the problem fail. We instead rely on a continuity argument for the approximating Lane-Emden problems

$$\begin{cases} -\Delta u = \dfrac{2}{q-1}\,(u^q - u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

as $q \to 1^+$. This is a joint work with M. Squassina and M. Gallo (Catholic University of Brescia).

References

[1] O. Alvarez, J.-M. Lasry, and P.-L. Lions, *Convex viscosity solutions and state constraints*, J. Math. Pures Appl. 76 (1997), 265–288.

[2] A.U. Kennington, *Power concavity and boundary value problems*, Indiana Univ. Math. J. 34 (1985), 687–704.

Simon Nowak (Bielefeld)

Nonlinear nonlocal potential theory at the gradient level

We present pointwise gradient potential estimates for a class of nonlinear nonlocal equations related to quadratic nonlocal energy functionals. Our pointwise estimates imply that the first-order regularity properties of such general nonlinear nonlocal equations coincide with the sharp ones of the fractional Laplacian. The talk is based on joint work with Lars Diening (Bielefeld), Kyeongbae Kim (Seoul) and Ho-Sik Lee (Bielefeld).

Jihoon Ok (Seoul)

Mean oscillation condition on nonlinear equations and regularity results

We consider general nonlinear elliptic equations of the form

$$\operatorname{div} A(x, Du) = 0 \quad \text{in } \Omega,$$

where $A : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies a quasi-isotropic (p, q)-growth condition, which is equivalent the pointwise uniform ellipticity of $A(x, \xi)$ under a suitable (p, q)-growth condition. We establish sharp and comprehensive mean oscillation conditions on $A(x, \xi)$ with respect to the x variable to obtain C^1- and $W^{1,\gamma}$-regularity results. The results provide new conditions, even in special cases such as $A(x, \xi) = a(x)|\xi|^{p-2}\xi$ and $A(x, \xi) = |\xi|^{p(x)-2}\xi$. This is joint work with Peter Hästö from University of Helsinki and Mikyoung Lee from Pusan National University.

Antonella Passarelli (Napoli)

Widely degenerate parabolic problems

We present some regularity results for the gradient of the weak solutions to strongly degenerate parabolic PDE's that arise gas filtration problems. More precisely, we will present a continuity result of a suitable function of the gradient as well as a second order regularity result of a nonlinear function of the spatial gradient Du, which in turn implies the existence of the weak time derivative. The results are obtained in collaboration with V. Bogelein (Salzburg University), F. Duzaar (Salzburg University) and R. Giova (University of Naples "Parthenope") and with P. Ambrosio (University of Naples "Federico II") and A. Gentile (Università Politecnica delle Marche).

Mathias Schäffner (Halle-Wittenberg)

On homogenization in nonlinear elasticity

Homogenization aims at deriving effective models for functionals or PDEs with highly oscillating spatial dependence. A major problem is to understand homogenization in the context of nonlinear elasticity, where typically constraints on the determinant of the gradient pose a substantial problem. After a short introduction to the subject and discussing relations to functionals with nonstandard growth conditions, we show that the standard multi-cell formula yields an upper bound for the Gamma-limit for a large class of functionals modeling incompressible materials with periodic micro-structure. This is based on a recent joint work with Matthias Ruf. If time permits I will also discuss briefly some quantitative homogenization results in the context of (compressible) nonlinear elasticity.

Thomas Schmidt (Hamburg)

Perimeter and total variation functionals with measure data

The talk deals with functionals in the calculus of variations which are the sum of either a perimeter term (parametric setting) or a total variation term (non-parametric setting) and an m-volume term. Here, m is a possibly lower-dimensional signed measure which has the role of a given right-hand side in corresponding Euler equations. The aim is at proving existence of minimizers of such functionals via semicontinuity despite possible cancellation-compensation effects between the different terms. The decisive assumptions on the measure m turn out to be certain (small-volume) isoperimetric conditions, which admit a wide class of (n-1)-dimensional measures and are partially optimal and interesting in themselves. The general theory will be illustrated with various examples. The results in a non-parametric setting are based on joint work with E. Ficola (Hamburg).

Berardino Sciunzi (Calabria)

Symmetry and monotonicity results in the context of semilinear problems involving singular nonlinearities

I shall consider elliptic problems addressing the study of the geometric properties of the solutions. This issue is in general related to the classification of the solutions or to Liouville type theorems. During the talk, in particular, I will discuss a recent classification result for the solutions to semilinear elliptic problems in half spaces involving singular nonlinerarities.

Susanna Terracini (Torino)

Singularly perturbed elliptic systems modeling partial separation and their free boundaries

We investigate the asymptotic behavior, as $\beta \to +\infty$, of solutions to competition-diffusion system of type

$$\begin{cases} \Delta u_{i,\beta} = \beta u_{i,\beta} \prod_{j \neq i} u_{j,\beta}^2 & \text{in } \Omega, \\ u_{i,\beta} = \varphi_i \geq 0 & \text{on } \partial\Omega, \end{cases} \qquad i = 1, 2, 3,$$

where $\varphi_i \in W^{1,\infty}(\Omega)$ satisfy the *partial segregation condition*

$$\varphi_1 \, \varphi_2 \, \varphi_3 \equiv 0 \quad \text{in } \overline{\Omega}.$$

For $\beta > 1$ fixed, a solutions can be obtained as a minimizer of the functional

$$J_\beta(\mathbf{u}, \Omega) := \int_\Omega \Big(\sum_{i=1}^{3} |\nabla u_i|^2 + \beta \prod_{j=1}^{3} u_j^2 \Big) \, dx$$

on the set of functions in $H^1(\Omega, \mathbb{R}^3)$ with fixed traces on $\partial\Omega$. We prove *a priori* and *uniform in* β Hölder bounds. In the limit, we are lead to minimize the energy

$$J\mathbf{u}, \Omega) := \int_\Omega \sum_{i=1}^{3} |\nabla u_i|^2 \, dx$$

over all partially segregated states:

$$u_1 \, u_2 \, u_3 \equiv 0 \quad \text{in } \overline{\Omega}$$

satisfying the given, partially segregated, boundary conditions above. We prove regularity of the free boundary up to a low-dimensional singular set. This is joint work with Nicola Soave.

Contributed talks

Carlo Alberto Antonini (Firenze)
Gradient regularity for quasilinear operators of mixed local non-local type

In this talk, we deal with mixed local-nonlocal quasilinear operators, modeled upon the sum of a p-Laplacian and a fractional (s, q)-Laplace operator. We address local and global gradient regularity for solutions under suitable assumptions on the parameters $p, q > 1$, $s \in (0, 1)$, the right-hand side, the reference domain Ω, and the outer data in the case of Dirichlet problems.

Linus Behn (Bielefeld)
Nonlocal equations with degenerate weights

We give a definition for fractional weighted Sobolev spaces with degenerate weights. We provide embeddings and Poincare inequalities for these spaces and show robust convergence when the parameter of fractional differentiability goes to 1. Moreover, we prove local Hölder continuity and Harnack inequalities for solutions to the corresponding weighted nonlocal integrodifferential equations. Joint work with Lars Diening (Bielefeld), Jihoon Ok (Seoul), and Julian Rolfes (Bielefeld).

Filomena De Filippis (Parma)
Schauder estimates at nearly linear growth

Variational integrals at nearly linear growth appear in the theory of plasticity with logarithmic hardening, that is the borderline configuration between plasticity with power hardening and perfect plasticity. The related (very challenging) regularity theory for minima has been intensively developed over the last 25 years, see e.g. the work of Frehse & Seregin ’99, Fuchs & Mingione ’00, Bildhauer ’03, Beck & Schmidt ‘13, Beck & Bulíček & Gmeineder ’20, Di Marco & Marcellini ’20, Gmeineder & Kristensen ’22, De Filippis & Mingione ’23. In this respect, we will discuss an intrinsic approach to the theory of Schauder for general nonautonomous functionals at nearly linear growth that covers the most common model examples in the literature. From recent, joint work with Cristiana De Filippis (Parma), Peter Hästö (Helsinki), and Mirco Piccinini (Pisa).

Ho-Sik Lee (Bielefeld)

Nonlocal Meyers' example

We consider examples of solutions to the nonlocal equation with irregular coefficients, which tell us that the higher differentiability and integrability of solutions are limited. Inspired by Meyers' example given in [Meyers '63] and [Kh. Balci, Diening, Giova, Passarelli di Napoli '22], we give nonlocal versions of such examples that are also robust when the order of the nonlocal equation converges to 2. This is the joint work with Prof. Anna Kh. Balci (Charles), Prof. Lars Diening (Bielefeld), and Prof. Moritz Kassmann (Bielefeld).

Mirco Piccinini (Parma)

Harnack inequalities for kinetic integral equations

We investigate local properties of weak solutions to a wide class of kinetic equations where the diffusion term in velocity is an integro-differential operator, having nonnegative kernel of fractional order with merely measurable coefficients. Under sufficient integrability along the transport variables on the nonlocal tail we provide an explicit local boundedness estimate, and also we show how to deduce from such result an Harnack inequality. This is based on a joint project in collaboration with F. Anceschi and G. Palatucci.

Sergio Scalabrino (Trieste)

Homogenization of non-local energies on disconnected sets

We consider the problem of the homogenization of non-local quadratic energies defined on δ-periodic disconnected sets defined by a double integral, depending on a kernel concentrated at scale ε. For kernels with unbounded support we show that we may have three regimes:

(i) $\varepsilon << \delta$, for which the Γ-limit even in the strong topology of L^2 is 0;

(ii) $\varepsilon/\delta \to \kappa$, in which the energies are coercive with respect to a convergence of interpolated functions, and the limit is governed by a non-local homogenization formula parameterized by κ;

(iii) $\delta << \varepsilon$, for which the Γ-limit is computed with respect to a coarse-grained convergence and exhibits a separation-of-scales effect; namely, it is the same as the one obtained by formally first letting $\delta \to 0$ (which turns out to be a pointwise weak limit, thanks to an iterated use of Jensen's inequality), and then, noting that the outcome is a nonlocal energy studied by Bourgain, Brezis and Mironescu, letting $\varepsilon \to 0$.

A slightly more complex description is necessary for case (ii) if the kernel is compactly supported. This is a joint work with A. Braides and C. Trifone

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Jule Schütt (Hamburg)

Optimal partial regularity for sets with variational mean curvature in L^p

A function $H \in \mathrm{L}^1(U)$ is called variational mean curvature of the set E in the open set U if E locally minimizes the Massari-functional

$$F \mapsto \mathrm{P}(F, U) - \int_{F \cap U} H.$$

The corresponding regularity theorem of Massari provides a strong connection between the integrability of H, i.e. $H \in \mathrm{L}^p$ for $p > n$, and the $\mathrm{C}^{1,\alpha}$-regularity of $\partial^* E$. This, in fact, is far away from being trivial in view of the lack of continuity of H. Massari conjectured that the optimal Hölder exponent α_{opt} converges to 1 as $p \to \infty$. We confirm this conjecture by presenting the exact formula for α_{opt} as a consequence of related partial regularity theory for variational integrals with Morrey-Hölder zero-order terms.