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Endogenous Growth Cycles—Arrow's Learning by Doing Reconsidered*

A two-sector growth model is presented in which human capital is acquired through learning by doing. It is shown that, for both the competitive situation and the social optimum, endogenous growth cycles may be the outcome. Concerning the economic prerequisite for persistent oscillations we detect a bunching of investment at nearby dates leading economic variables to overshoot the long-run steady state values. This clustering of investment, for its part, may be caused by adjacent complementarity with respect to the stocks or by a sufficiently strong external effect of investment on the marginal product of physical capital or on the opportunity costs of investment.

1. Introduction

The importance of human capital for long-run economic growth has been increasingly stressed in a great many publications during the last few years. As to the formation of the stock of human capital, it is generally agreed upon that it is not given exogenously, but determined endogenously. Basically, two different approaches can be distinguished.

On the one hand, there are growth models in which human capital is built by explicitly devoting time to its formation (Lucas 1988; Laitner 1993; Caballe and Santos 1993). In these models, agents permanently increase their stock of knowledge by deciding how much to work and how much to learn. Thus, human capital can grow without an upper bound, leading to sustained per capita growth of economic variables.

On the other hand, growth models exist in which positive externalities of physical capital, which are external to the firms, lead to increasing returns on a macroeconomic level thus generating sustained per capita growth (see e.g. Romer 1986). This approach goes back to Arrow (1962), who found out that acquiring new knowledge is strongly related to experience. For example, he refers to the airframe industry where a strong correlation between productivity growth and experience seems to exist. A measure for the change in experience may be seen in investment, and Arrow maintains that cumulative

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investment therefore represents a good index for the stock of human capital. Assuming that knowledge is formed as a by-product of investment also makes this variable endogenous.

These models, however, are primarily interested in explaining long-run per-capita growth and, therefore, confine their analysis to balanced growth paths, that is, dynamic paths on which all economic variables grow at the same rate. Moreover, although the number of papers presenting endogenous growth models has sharply increased during recent years, the dynamics of such models are not yet well understood (cf. Caballe and Santos 1993, 1043). This holds although there have emerged papers studying transitional dynamics in growth models with human capital (see e.g., Mulligan and Sala-i-Martin 1993 or King and Rebelo 1993). Other papers which give an explicit analysis of the dynamics of growth models are the ones by Boldrin and Rustichini (1994), Chamley (1993), Benhabib and Perli (1994), or Xie (1994). These authors examine their models as to the emergence of multiple steady states or indeterminacy of equilibrium paths.

Another line of research was initiated by Benhabib and Nishimura (1979). These authors demonstrate, applying the Hopf-Bifurcation theorem, that n-sector growth models may, under certain conditions, reveal persistent oscillations. In this model certain factor intensities turn out to be necessary for growth cycles (see for example Nishimura and Takahashi 1992). But the analysis crucially depends on the fact that they only consider tangible capital goods which have a competitive price and their result is only valid for that sort of model and cannot be applied to models with human capital which do not have this property. Our goal with this paper, therefore, is to present a two-sector model of economic growth in which human capital is formed according to the learning by doing approach, and derive economic conditions which may prevent the system from converging to a steady state and instead bring about endogenous cycles.

As to the empirical relevance of growth cycles, there has been a lively debate whether they exist or not. But this question seems to be very difficult to handle because little is known about the statistical reliability of the time series which are tested the further we go back in time. But nevertheless, as Rosenberg and Frischtak state, "No one . . . can doubt that they [dynamics of capitalist economies] experience significant long-term variations in their aggregate performance. The question is whether these long-term variations are more than the outcome of a summation of random events" (Rosenberg and Frischtak 1983, 146).

The rest of the paper is organized as follows. In the next section, we present our model and derive necessary optimality conditions both for the competitive situation and the social optimum. In Section 3, we study the dynamic behavior of our model near the steady state and give conditions which must be fulfilled for permanent oscillations of the variables. Section 4 illustrates our analytical results with the help of a numerical example, and Section 5 finally concludes the paper.

2. A Two-Sector Growth Model with Learning by Doing

The starting point in models of growth theory is often the assumption of a single representative individual with household production whose goal consists in maximizing a discounted stream of utility arising from consumption $F[C(t)] \equiv \int_0^\infty e^{-\rho t} u(C(t))\, dt$, where ρ denotes the discount factor, $u(\cdot)$ is a concave utility function, and $C(t)$ stands for the consumer good. The constraint this individual has to obey is given by a differential equation of the form $\dot{K}(t) = I(t) - \delta K(t)$, with $K(t)$ stock of physical capital, $I(t)$ investment and δ depreciation rate.[1] The decision problem consists in determining what amount to consume and how much to invest and, thus, to increase the consumption possibilities in the future.

In addition to the stock of physical capital which is formed as a result of investment we postulate that there are spillover effects of investment which consist in building up a stock of human capital, denoted as $A(t)$. More concretely, we suppose that $A(t)$ reflects cumulated experience in the production of investment goods, that is, $A(t)$ is built up according to Arrow's learning by doing approach, Arrow (1962). In contrast to Arrow, however, who uses a vintage approach with fixed coefficients, in our model technical progress is disembodied and the production function is not restricted to fixed coefficients (see Levhari 1966). Moreover, we suppose that the contribution of gross investment to the formation of human capital further back in time is smaller than recent gross investment.

This assumption makes sense economically and can be formalized by defining the stock of human capital as an integral of past gross investment with exponentially declining weights put on investment flows further back in time (cf. Ryder and Heal 1973 or Feichtinger and Sorger 1988). $A(t)$ then is given by $A(t) = \varphi \int_{-\infty}^{t} e^{\varphi(s-t)} I(s)ds$. The parameter φ represents the weight given to more recent levels of gross investment. The higher φ, the larger is the contribution of more recent gross investment to the human capital stock in comparison to flows of investment dating back further in time.

Furthermore, supposing a two-sector economy, the consumer good can be expressed as a function depending on the stock of human capital $A(t)$, the stock of physical capital, and the investment good, i.e. $C(t) = T(A(t),K(t),I(t))$. This function denotes the production possibility frontier (PPF) and is formally obtained by solving the static optimization problem of the consumer

[1]The dot over a variable denotes the derivative with respect to time.

good sector and the investment good sector respectively, and inserting the resulting inputs in the production function of the investment good sector. For a more detailed treatment of PPFs we refer to Boldrin (1989) or Benhabib and Nishimura (1979).

The PPF[2] $T(A,K,I)$ in our model is assumed to be C^2, to have $T_I(\cdot) < 0$, and to be strictly concave in I. Furthermore, it is increasing in the factors A and K with decreasing marginal productivities and has an upper bound for each marginal product.

The dynamic optimization problem of the representative individual then consists in maximizing $F[T(A,K,I)]$ with respect to I subject to the constraint giving the evolution of physical capital. The evolution of the stock of human capital, however, is not taken into account by the representative individual. But this turns out to be non-optimal since it becomes intuitively clear that by only taking into account the effects of investment on the creation of physical capital neglects the positive effects of investment on the creation of human capital. It should also be noticed that the solution to this optimization problem is equivalent to the solution of the competitive equilibrium. A formal proof can be obtained by adopting the arguments in the paper by Becker (1981). In what follows we refer to the situation where those spillovers are neglected as the competitive situation.

The social optimum will be called the solution where those positive externalities of investment are intentionally taken account of. Formally, this can be achieved by considering an additional constraint in the individual's optimization problem. From above we know that the stock of human capital is formed as a by-product of accumulated weighted gross investment, $A(t) = \varphi \int_{-\infty}^{t} e^{\varphi(s-t)} I(s) ds$. The evolution of $A(t)$ is then given by the differential equation $\dot{A}(t) = \varphi(I(t) - A(t))$ and the social optimal solution for the representative individual consists in maximizing his discounted stream of consumption subject to both the constraint giving the evolution of physical capital as well as the evolution of the stock of human capital.

Summarizing our considerations from above, the competitive situation of our economy can be described by a solution to the optimization problem (I):

$$\max_{\{I(t)\}} \int_0^\infty e^{-\rho t} U(A(t), K(t), I(t)) dt \,,$$

with $U(A(t),K(t),I(t)) = u(T(A(t),K(t),I(t)))$ and subject to $\dot{K}(t) = I(t) - \delta K(t)$, $K(0) = K_0 > 0$.

The social optimum can be described by a solution to our problem (II)

[2]In the following we will suppress the time argument where it is dispensable.

$$\max_{\{I(t)\}} \int_0^\infty e^{-\rho t} U(A(t), K(t), I(t)) dt \, ,$$

with $U(A(t),K(t),I(t)) = u(T(A(t),K(t),I(t)))$ and subject to $\dot{K}(t) = I(t) - \delta K(t)$, $K(0) = K_0 > 0$ and $\dot{A}(t) = \varphi(I(t) - A(t))$, $A(0) = A_0 > 0$.

Before we go on and use necessary conditions to characterize an optimal solution, we show that a solution to our optimization problem exists. This is done in Theorem 1.

THEOREM 1. *Given the assumption of strict concavity of $U(A,K,I)$ in I, there exists a unique path of investment that solves the optimal control problem (I) and the optimal control problem (II).*

The proof, which is available on request in an appendix, follows from a standard result in control theory (cf. Seierstad and Sydsaeter 1987, 237) and uses the fact that the domain of all possibly optimal values for the rate of investment is bounded. This is a consequence of our assumption that the marginal product of each factor in the PPF is bounded by above.

Given Theorem 1 we can now characterize the solution to our optimization problems. First let us look at problem (I), the competitive situation. The Hamiltonian function for that problem then is given by the expression $H(\cdot) = \gamma_0 U(A,K,I) + \gamma_1(I - \delta K)$, with γ_1 denoting the current-value co-state variable or shadow price of capital. The first-order condition for $I(t)$ to yield a maximum for problem (I) then is $-U_I(\cdot) = \gamma_1$ (for $\gamma_0 = 1$). The evolution of γ_1 is given by $\dot{\gamma}_1 = (\rho + \delta)\gamma_1 - U_K(\cdot)$. Here, it should be noted that the stock of human capital also evolves over time, as a by-product of investment, and thus influences the evolution of physical capital, of its shadow price and of investment. But this property is not explicitly taken into account by our individual. Furthermore, the limiting transversality conditions are given by $\lim_{t\to\infty} e^{-\rho t}\gamma_1(t)K(t) = 0$. Note that the transversality conditions are necessary in this case (demonstrated in the appendix available on request). This result follows from Michel's corollary to his theorem (Michel 1982, 977–79; see also Seierstad and Sydsaeter 1987, 244–55). Before analyzing the dynamic behavior of our variables let us briefly turn to the social optimization problem, denoted as problem (II).

The only difference to problem (I) consists in the fact that in this problem the individual that may be termed a social planner takes account of the positive spillovers of investment. The Hamiltonian function is now written as $H(\cdot) = \gamma_0 U(A,K,I) + \gamma_1(I - \delta K) + \gamma_2\varphi(I - A)$, with γ_2 denoting the shadow price of human capital. Note that γ_2 only represents a shadow price, whereas γ_1 represents the competitive price of the investment good as well (see e.g. Boldrin 1989, 235). The rate of investment is now set according to $-U_I(\cdot) = \gamma_1 + \varphi\gamma_2$. Note that for interior solutions to problem (I) and (II), which may be justified by imposing Inada-type conditions on $T(\cdot)$, it can easily be seen that γ_0 can be set equal to 1. It can already be seen that in problem (II) investment is always larger than in problem (I). The reason is that now investment is not only paid its competitive price γ_1 but also an additional weighted (shadow) price $\varphi\gamma_2$, giving the value of an additional marginal unit of human capital. The dynamic behavior of γ_2 is described by $\dot{\gamma}_2 = (\rho + \varphi)\gamma_2 - U_A(\cdot)$. The limiting transversality conditions for that case are given by $\lim_{t\to\infty} e^{-\rho t}(\gamma_1(t)K(t) + \gamma_2(t)A(t)) = 0$.

In the next section, we will investigate the dynamic behavior and try to give economic conditions for a possibly cyclical behavior of the variables.

3. The Dynamic Behavior—Analytical Results

The Competitive Economy

Let us first look at the competitive economy. We know that the evolution of physical capital and human capital is described by the differential equations $\dot{K}(t) = I(t) - \delta K(t)$ and $\dot{A}(t) = \varphi(I(t) - A(t))$, respectively. Investment in these two equations is chosen so that $H(\cdot) = \gamma_0 U(A,K,I) + \gamma_1(I - \delta K)$ is maximized, giving $-U_I(A,K,I) = \gamma_1$, as already mentioned in the last section. Investment $I(t)$ is thus a function implicitly defined by A,K,γ_1, that is, $I(t) = I(A(t),K(t),\gamma_1(t))$. As to the evolution of the shadow price $\gamma_1(t)$, we know from optimal control theory that it is given by the differential equation $\dot{\gamma}_i(t) = \rho\gamma_1(t) - \partial H(\cdot)/\partial K$, so that the system of differential equations can be written as

$$\dot{K}(t) = I(A(t), K(t), \gamma_1(t)) - \delta K(t)\,, \tag{1}$$

$$\dot{\gamma}(t) = (\rho + \delta)\gamma_1(t) - U_K(\cdot)\,, \tag{2}$$

$$\dot{A}(t) = \varphi I(A(t), K(t), \gamma_1(t)) - \varphi A(t)\,. \tag{3}$$

To determine the dynamic behavior of our system of differential equations, let us first investigate the question of the existence of a steady state in the sense of a rest point, that is, a situation where the derivatives with respect to time equal zero. Here we can state Lemma 1.

LEMMA 1. *Under a slight additional assumption, the system of differential equations (1)–(3) has a unique optimal steady state K^*,γ_1^*,A^*.*
The proof of this lemma is contained in the appendix available on request.

To derive the local dynamic behavior of the economic variables we now compute the Jacobian of (1)–(3) and determine its eigenvalues. The derivatives of $I(t) = I(A(t),K(t),\gamma_1(t))$ are easily obtained by implicit differentiation as $I_A(\cdot) = -U_{IA}/U_{II}$, $I_K(\cdot) = -U_{IK}/U_{II}$, $I_{Y1}(\cdot) = -1/U_{II} > 0$. Thus, the Jacobian can be seen to have the following form,

$$J = \begin{bmatrix} -U_{IK}/U_{II} - \delta & -1/U_{II} & -U_{IA}/U_{II} \\ -U_{KK} + U_{IK}^2/U_{II} & U_{IK}/U_{II} + \rho + \delta & -U_{KA} + U_{IK}U_{IA}/U_{II} \\ -\varphi U_{IK}/U_{II} & -\varphi/U_{II} & -\varphi - \varphi U_{IA}/U_{II} \end{bmatrix}.$$

The characteristic equation of our system is

$$\lambda^3 + (-\text{trace } J)\lambda^2 + W_2\lambda + (-\det J) = 0 \, ,$$

with W_2 being defined as

$$W_2 = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} ,$$

with a_{ij} element of the i-th row and j-th column of J.

As mentioned in the introduction, our goal with this paper is to study the out of steady-state dynamics of our growth model and especially to work out under what conditions endogenous growth cycles may appear. From the technical point of view, this may be achieved by applying the Hopf bifurcation theorem. The prerequisite for that phenomenon lies in the presence of two eigenvalues crossing the imaginary axes, thus causing a change in the qualitative property of the solution of the system of differential equations. For our system we can use Lemma 2 in order to gain further insight in the properties of our economic model.

LEMMA 2. *A necessary and sufficient condition for the characteristic equation $\lambda^3 + (-traceJ)\lambda^2 + W_2\lambda + (-detJ) = 0$ to possess a pair of two purely imaginary roots $\pm wi$, $i = \sqrt{-1}$, $w \neq 0$ is $W_2 > 0$ and $W_2 \cdot (-traceJ) + detJ = 0$.*

A proof of that Lemma can be found in Asada and Semmler (1995).

Applying this Lemma to our problem we have to find out that the second condition can be determined technically and becomes extremely complicated. Moreover, it is not apt to economic interpretation so that we will focus on the first one. Doing so, we calculate W_2 as $W_2 = -a + V_1 + c$, with $a = \delta(\rho + \delta) + \rho\varphi > 0$, $V_1 = (-1/U_{II})[(\rho + 2\delta)U_{IK} + U_{KK}]$, $c = -(\varphi/U_{II})(\rho U_{IA} + U_{KA})$.

It can immediately be seen that the elements W_2 is composed of may have positive or negative signs except a.

If we want to give an economic interpretation to these terms, we can first state that V_1 can be interpreted as a measure giving complementarity over time. If $V_1 > 0$ we can speak of complementarity between adjacent dates (or briefly adjacent complementarity) with respect to the capital stock K. That means increasing investment at time t_3 implies a reallocation of resources from distant dates t_1 to nearby dates t_2. Correspondingly, if $V_1 < 0$ we speak of complementarity between distant dates (distant complementarity) with respect to the capital stock, meaning that an increase in investment at time t_3 leads to a reallocation of resources from nearby dates to distant ones t_1.[3]

Looking at the constant W_2 in our model, we see that adjacent complementarity is a necessary condition for endogenous growth cycles if $c \leq 0$. What does this mean? If there is adjacent complementarity with respect to the capital stock, then an economy which intends to do a lot of investment at the end of a year would tend to invest little at the beginning of that year but much in the middle of the year, while with distant complementarity that economy would invest a lot at the beginning and relatively little in the middle of the year. In other words, adjacent complementarity favors investment at nearby dates leading to a clustering of investments. To investigate this mechanism further let us examine a country with an initially low stock of physical capital. To reach the steady-state level of capital, investment will increase. Given adjacent complementarity, however, this increment in investment leads to a further rise at nearby dates, which eventually makes this country achieve a higher capital stock than the steady-state level. This overshooting may be intensified by the economy getting used to low consumption so that there is no rush to raise it to the long-run optimal level as Ryder and Heal (1973, 15) formulated it. It should be noted that this overshooting is caused by adjacent complementarity of the physical capital stock and determined by the preferences and by the technology of our economy. If there is distant complementarity, no cycles are possible for $c \leq 0$.

The sign of c, however, may also be positive and, thus, lead to endogenous growth cycles even for distant complementarity of physical capital. Looking at c, we see that it is determined by the effect of an increase in human capital on the marginal products of physical capital and of investment in $U(\cdot)$. But the increase in human capital in our learning by doing model is nothing else than the external effect of an additional unit of investment. Therefore, we can state that the expression c is determined by the external effect of investment on the opportunity cost of investment and on the marginal product of capital in the PPF.

To be more precise, let us look a little closer at the expression c, which is given by $c = (-\varphi/U_{II})(\rho U_{IA} + U_{KA})$. The signs of $U_{KA}(\cdot)$ and $U_{IA}(\cdot)$ are obtained as $U_{KA}(\cdot) = u'(\cdot)T_{KA} + u''(\cdot)T_A(\cdot)T_K(\cdot)$ and $U_{IA}(\cdot) = u'(\cdot)T_{IA}(\cdot) + u''(\cdot)T_I(\cdot)T_A(\cdot)$. At the steady state, it must hold that $-T_I(\cdot)(\rho + \delta) = T_K(\cdot)$, so

[3]For a detailed derivation of these concepts, see Wan (1970), Ryder and Heal (1973), or Dockner and Feichtinger (1991).

that c may be written as $c = -(\varphi/U_{II})(\rho u'(\cdot)T_{IA} + u'(\cdot)T_{KA} - \delta T_I(\cdot)\, T_A(\cdot)u''(\cdot))$. This shows that c can only be positive if the external effect of investment (via an increase of human capital) shows strong positive effects on the marginal productivity of physical capital (if $T_{KA} > 0$) and/or strong positive effects on the opportunity cost of investment, that is if it reduces the opportunity costs (if $T_{IA} > 0$).

These considerations show that the cross derivatives of $U(\cdot)$ cannot be determined a priori and depend on the exact specification of the functions. However, it can clearly be stated what conditions must be fulfilled so that c is positive and, thus, growth cycles may occur. For c to be positive there must be a strong positive external effect of investment on the marginal productivity of physical capital in the PPF or on the opportunity cost of investment. If giving up consumption does not only raise physical capital but additionally also shows strong externalities at the margin, meaning that an increase in investment reduces the opportunity costs of investment via an increase in A and/or raises the marginal productivity of physical capital, then society is more willing to forgo consumption and instead make investments. This holds because our economy can thus not only increase the future capital stock but also raise the marginal productivity of capital and/or reduce the opportunity cost of investment. Thus investment becomes more profitable and/or cheaper. This mechanism may lead to a bunching of investment at nearby dates, causing the overshooting of the long-run steady state values just as in the case of adjacent complementarity with respect to the capital stock.

We can now summarize our results in the following Theorem.

THEOREM 2. *A necessary condition for persistent oscillations of the economic variables consists in (i) adjacent complementarity with respect to the physical capital stock or (ii) a strong positive external effect of investment on the marginal product of physical capital or investment.*

The proof of that theorem follows from Lemma 2 together with the characterization of W_2.

It should be noted that this theorem only provides us with necessary conditions for persistent growth cycles. Because of those reasons, later on, we will present a numerical example to illustrate our results.

Up to now we have derived results for our competitive economy and have seen that, under certain conditions, endogenous growth cycles may be the outcome. As usual, in this sort of model the social optimum does not coincide with the competitive solution so that policy makers have to give incentives for investment. They can do this by imposing taxes on consumption which may then be used to subsidize investment. However, the question that remains is what time paths may be the outcome in the social optimum. This question will be investigated in the next subsection.

The Social Optimum

As already mentioned in the previous section, for the social optimum maximization problem the rate of investment is at any point of time higher than in the competitive economy. The maximum principle now gives $-U_I(A,K,I) = \gamma_1 + \varphi\gamma_2$, implicitly defining investment. The derivatives can again be calculated as $I_A(\cdot) = -U_{IA}/U_{II}$, $I_K(\cdot) = -U_{IK}/U_{II}$, $I_{\gamma 1}(\cdot) = -1/U_{II} > 0$ and $I_{\gamma 2}(\cdot) = -\varphi/U_{II} > 0$. Substituting these relations in the other necessary conditions given by the differential equations describing the evolution of physical capital, the stock of human capital and its shadow prices, respectively, then gives the so-called Hamiltonian system, completely describing the dynamic behavior of our economic variables. This system may be written as

$$\dot{K}(t) = I(A(t), K(t), \gamma_1(t), \gamma_2(t)) - \delta K(t)\,, \tag{4}$$

$$\dot{A}(t) = \varphi I(A(t), K(t), \gamma_1(t), \gamma_2(t)) - \varphi A(t)\,, \tag{5}$$

$$\dot{\gamma}_1(t) = (\rho + \delta)\gamma_1(t) - U_K(\cdot)\,, \tag{6}$$

$$\dot{\gamma}_2(t) = (\rho + \varphi)\gamma_2(t) - U_A(\cdot)\,. \tag{7}$$

Before going into the details of our analysis we state Lemma 3.

LEMMA 3. *Under a slight additional assumption, the canonical system (4)–(7) possesses a unique optimal steady state* $K^*, A^*, \gamma_1^*, \gamma_2^*$.

The proof of this lemma is also available in the appendix on request.

As in the preceding section we now calculate the Jacobian matrix near the steady state and determine its eigenvalues. The Jacobian is seen to be

$$J = \begin{bmatrix} -U_{IK}/U_{II} - \delta & -U_{IA}/U_{II} & -1/U_{II} & -\varphi/U_{II} \\ -\varphi U_{IK}/U_{II} & -\varphi U_{IA}/U_{II} - \varphi & -\varphi/U_{II} & -\varphi^2/U_{II} \\ -U_{KK} + U_{KI}^2/U_{II} & -U_{KA} + \dfrac{U_{KI}U_{AI}}{U_{II}} & \rho + \delta + U_{IK}/U_{II} & \varphi U_{KI}/U_{II} \\ -U_{KA} + \dfrac{U_{KI}U_{AI}}{U_{II}} & -U_{AA} + U_{AI}^2/U_{II} & U_{IA}/U_{II} & \rho + \varphi + \varphi U_{IA}/U_{II} \end{bmatrix}$$

with the eigenvalues given by

$$\lambda_{1,2,3,4} = \frac{\rho}{2} \pm \sqrt{\left(\frac{\rho}{2}\right)^2 - \frac{W_1}{2} \pm \sqrt{\left(\frac{W_1}{2}\right)^2 - \det J}}\,.$$

W_1 is defined as

$$W_1 = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{22} & a_{24} \\ a_{42} & a_{44} \end{vmatrix} + 2 \begin{vmatrix} a_{12} & a_{14} \\ a_{32} & a_{34} \end{vmatrix},$$

with a_{ij} again denoting the element of the ith row and jth column of J (see Dockner and Feichtinger 1991).

Given the explicit characterization of the eigenvalues of our Jacobian matrix, we can use Lemma 4 in order to investigate whether persistent cycles may also be possible in the social optimum.

LEMMA 4. *The conditions* $\det J > (W_1/2)^2$ *and* $\det J - (W_1/2)^2 - \rho^2(W_1/2) = 0$ *are necessary and sufficient for all eigenvalues to be complex and two having zero real parts.*
The proof of that Lemma can be found in Dockner and Feichtinger (1991).

For our system the constant W_1 and the determinant of the Jacobian can be written as follows:

$$W_1 = -a - b + V_1 + V_2 - 2\varphi U_{KA}/U_{II},$$

$$\det J = ab - aV_2 - bV_1 + \varphi U_{KA}/U_{II}[\delta(\rho + \varphi) + \varphi(\rho + \delta)],$$

$$\det J - (W_1/2)^2 = 1/4[-(a - b)^2 - (V_1 + V_2)^2 - 2(a - b)(V_2 - V_1)]$$
$$+ \varphi\frac{U_{KA}}{U_{II}}[\delta(\rho + \varphi) + \varphi(\rho + \delta)] + \varphi\frac{U_{KA}}{U_{II}}\ W_1 + \left(\varphi\frac{U_{KA}}{U_{II}}\right)^2,$$

where $a = \delta(\rho + \delta)$, $b = \varphi(\rho + \varphi)$, $V_1 = -(1/U_{II})\,[(\rho + 2\delta)U_{IK} + U_{KK}]$, $V_2 = -(\varphi/U_{II})\,[(\rho + 2\varphi)U_{IA} + U_{AA}]$.

It can already be seen that, as in the competitive situation, two purely imaginary eigenvalues may occur, but again it is not possible to give sufficient conditions leading to that phenomenon because of the complexity of the constant W_1 and of the determinant of the Jacobian. Nevertheless, we can give necessary conditions which may be interpreted economically.

To do so, we first consider our model for the case $U_{KA} = 0$. Given Lemma 4 we see that both $W_1 > 0$ and $\det J - (W_1/2)^2 > 0$ are necessary conditions for persistent growth cycles. For W_1 to be positive at least one of the stocks must show adjacent complementarity and the degree of adjacent complementarity has to be sufficiently high. If one stock is characterized by distant complementarity while the other has adjacent complementarity, we see from the condition $\det J - (W_1/2)^2 > 0$ that the stock with the higher depreciation rate must show adjacent complementarity. If both stocks have adjacent complementarity the one with the higher depreciation rate must have a higher degree of complementarity than the one with the lower depreciation rate.

In contrast to the competitive situation we must now be aware that the social planner intentionally takes account of the positive spillover effects of investment. Therefore, the degree of complementarity with respect to the stock of human capital must also be taken into account in determining the conditions under which cycles may turn out to be optimal.

It should be recalled that we have derived these results with the assumption that $U_{KA} = 0$. Before we give up this assumption we summarize our results in Theorem 3.

THEOREM 3. *Given $U_{KA}(\cdot) = 0$ the following turns out to be true: a necessary condition for persistent endogenous growth cycles is (i) that the stock with the higher depreciation rate shows adjacent complementarity while the other has distant complementarity or (ii) that the degree of complementarity of the stock with the higher depreciation rate is larger than the one of the other stock if both stocks have adjacent complementarity.*

Let us now investigate what may happen if both stocks have distant complementarity but investment shows a large external effect influencing the marginal productivity of capital in the PPF. Then, just as in the competitive economy, both W_1 and $\det J$ may become positive so that endogenous growth cycles are possible. This holds because the sign of $U_{KA}(\cdot)$ may be positive if $T_{KA}(\cdot) > 0$. Thus, we can state THEOREM 4.

THEOREM 4. *If both stocks have distant complementarity, a necessary condition for persistent endogenous growth cycles is a sufficiently strong external effect of investment on the marginal productivity of physical capital in the PPF.*

Up to now we have shown for our two-sector economy[4] that a clustering of investments may lead economic variables to overshoot the long-run steady-state values and generate sustained cycles if certain conditions are met. However, it was not possible to derive sufficient conditions within the analytical model. In the next section, we will therefore present some numerical examples to illustrate our analytical findings.

4. A Numerical Example

As mentioned above, in this section we will illustrate our analytical results with a numerical example. To do this, we study a two-sector economy with a linear utility function, $u(C) = C$. The PPF is given by $T(A,K,I) = a_1K + a_2A - a_3I^2 / (A + K) + b_1IK + b_2IA$. The evolution of the capital stock is given by $\dot{K} = I - \delta K$. For the parameter values we choose $a_3 = 1, a_1 = 0.35$, $a_2 = 1$, $b_2 = -0.25$. ρ, δ and φ per period are set to $\rho = 0.25$, $\delta = 0.75$ and $\varphi = 0.5$. Taking data aggregated in 5-year periods means that the annual

[4]It can be shown that for a one-sector economy similar conditions may lead to endogenous cycles.

discount rate is 5%, the depreciation rate per year 15% and the annual φ is $\varphi_a = 0.1$. The value $\varphi_a = 0.1$ states that the contribution of investment five years back to the actual stock of human capital is $\exp(-5 \cdot \varphi_a) = 0.6065$.

Using the parameter values from above and taking b_1 as bifurcation parameter we see that for $b_{1,crit} = 0.16541$, two eigenvalues of the Jacobian matrix for system (1)–(3) are purely imaginary. The steady states for this value of b_1 are given by $K^* = 3.17228$, $\gamma_1^* = 0.9272186$, $A^* = 2.37921$, $I^* = 2.37921$, $C^* = 2.3031218$, $GNP^* = 4.5091696$. $GNP(t)$ denotes gross national product and is given by $GNP(t) = C(t) + \gamma_1(t)I(t)$. Note that $\gamma_1(t)$ denotes the price of investment in terms of the consumer good which is used as numeraire. It should also be mentioned that for these parameter values and the steady state values the PPF fulfills all of the properties required in the analytical part. The derivative of the real part with respect to the bifurcation parameter of the purely imaginary eigenvalues at $b_1 = b_{1,crit}$, is $Re\lambda_1'(b_{1,crit}) = 4.8916$ indicating the emergence of a Hopf bifurcation.[5]

By varying b_1 we determine the sign of V_1 giving the degree of complementarity of the capital stock with respect to time. As to the degree of adjacent complementarity, we calculate for V_1, $V_1 = 1.36981$ whereas both $-a$ and c are negative, for $b_1 = b_{1,crit}$. Taking $b_1 = 0.16$ a little smaller than $b_{1,crit}$, we calculate the eigenvalues of the Jacobian as $\lambda_{1,2} = -0.0291483 \pm 0.626392\,i$, $\lambda_3 = -0.359898$ indicating that for this case the dynamic behavior of the variables is characterized by a stable focus, with the path converging to the steady state in the long run. If we take b_1 a little larger than 0.16541 and take $b_1 = 0.16542$ we can observe that we now have stable limit cycles. In Figure 1 it can be seen how the trajectory approaches the limit cycle in the three dimensional $(I(t) - GNP(t) - A(t))$ phase diagram, demonstrating that it is an attractor. Let us next present a numerical example demonstrating the possibility of endogenously generated growth cycles for the social optimum.

Once more, we suppose a linear utility function and the PPF for the social optimization problem is again assumed to be given by $T(A,K,I) = a_1K + a_2A - a_3I^2/(A + K) + b_1IK + b_2IA$. The evolution of the capital stock is given by $\dot{K} = I - \delta K$ and human capital follows $\dot{A} = \varphi(I - A)$. For the parameter values we now choose $a_3 = 0.15$, $a_1 = 3.2$, $a_2 = 2.175$, $b_2 = 3.0855$. The annual discount rate and depreciation rate is now set to $\rho = 0.25$ and $\delta = 0.035$. φ per year is given by $\varphi = 0.15$. Again, b_1 is selected as bifurcation parameter. Forming the Hamiltonian (taking explicitly the constraint $\dot{A} = \varphi(I - A)$ into consideration), maximizing with respect to I and substituting this value in the differential equations then yields the modified Hamiltonian system (4)–(7).

[5]For the numerical computations and the solution of the differential equations we used the computer software Mathematica (see Wolfram Research 1991).

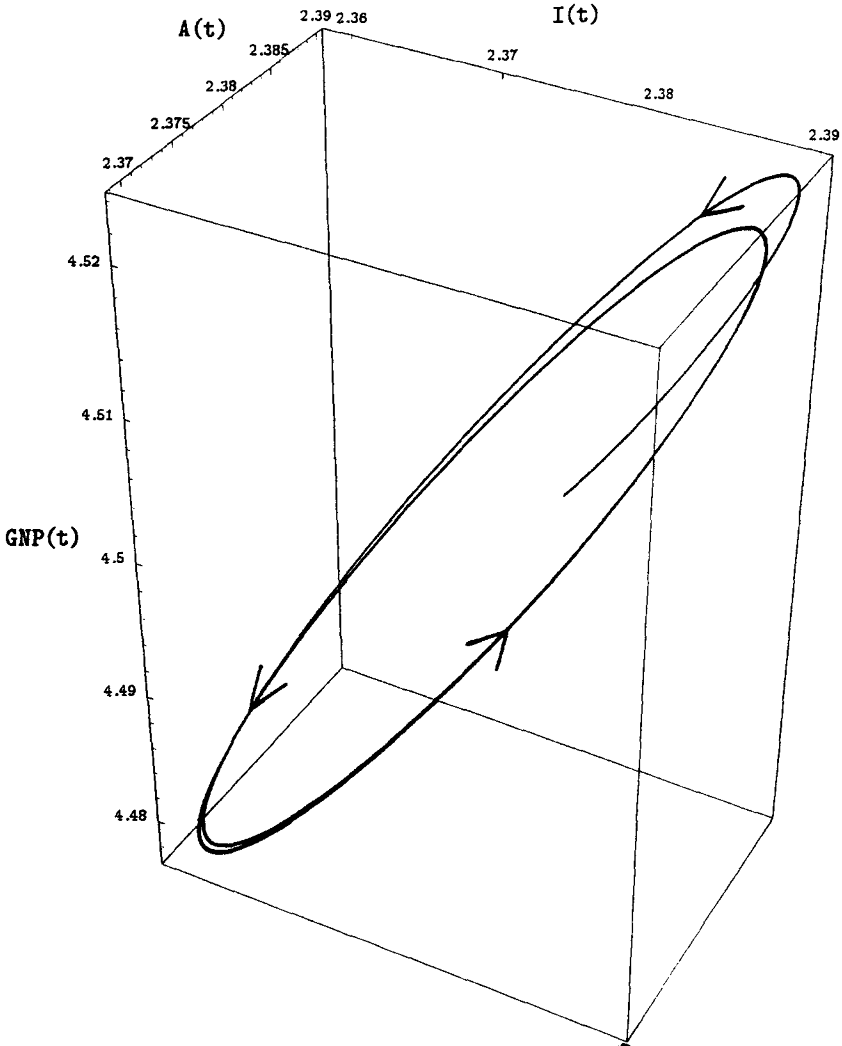

Figure 1.

To investigate this dynamic system we used the code BIFDD.[6] It turns out that this system has two purely imaginary eigenvalues for $b_1 = -0.712938$. The derivative of the real part of the purely imaginary eigenvalues with respect to b_1 at $b_1 = b_{1,crit}$ is given by $Re\lambda_1'(b_{1,crit}) = 24.73895$. BIFDD also

[6]For a description of the related code BIFOR2 we refer to Hassard, Kazarinoff, and Wan (1981).

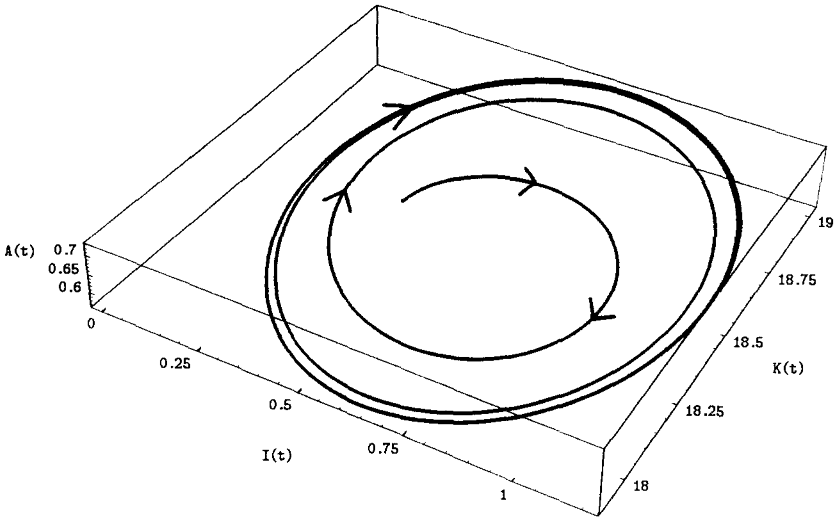

Figure 2.

calculates the coefficient β_2 determining the stability of the limit cycles which is given by $\beta_2 = -12.99039$. As $\beta_2 < 0$ the limit cycles are stable. The steady-state values for these parameters are now seen to be $K^* = 18.46036$, $A^* = 0.6461126$, $\gamma_1^* = 9.612614$, $\gamma_2^* = 10.42188$, $I^* = 0.6461126$, $C^* = 53.2597$, $GNP^* = 59.470531$. Again, it can be demonstrated that the PPF fulfills all of the properties required in the analytical part.

Given this information we can solve our system of differential equations. For slightly larger values of b_1 than $b_{1,crit}$ we again can observe stable limit cycles.

Figure 2 shows how the trajectory approaches the limit cycle in the three dimensional $(I(t) - A(t) - K(t))$ phase diagram, with $b_1 = -0.7129$.

As to the economic mechanisms for our numerical example we see that there is distant complementarity with respect to physical capital, with $V_1 = -14.5202$, and adjacent complementarity with respect to human capital, with $V_2 = 16.2148$ for $b_1 = b_{1,crit}$ and the corresponding parameter values. Note that b_1 again influences the value of V_1, thus determining the degree of complementarity of the stock of capital over time. As to the cross derivative T_{KA} we see that it is negative in the steady state, but extremely small, namely $T_{KA} = -1.79554 \cdot 10^{-5}$ so that this effect can be neglected.

5. Conclusion

In this paper we have demonstrated that a basic two-sector growth model with learning by doing may lead to persistent cycles if it is assumed that investment at different dates has different weights concerning its contribution to the stock of human capital, which is certainly reasonable. The conditions responsible for that phenomenon are intuitively plausible and have a nice economic interpretation.

Thus, we have seen that a model of economic growth may have richer dynamics than just monotonic convergence to a balanced growth path. But the emergence of persistent growth cycles is only one aspect. For example, Lemmas 1 and 3 in the paper point to the possibility of multiple steady states. A more thorough analysis of this aspect, however, was beyond the scope of this paper. Besides multiple steady states, the possible indeterminacy of equilibrium paths would also be worth investigating. Therefore, this paper underlines the necessity of further studies examining transitional dynamics in growth models along the line of the papers in the special *JET* volume (see *Journal of Economic Theory*, Vol. 63, No. 1, 1994).

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Appendix

List of Symbols

C	=	consumption.
I	=	investment.
A	=	human capital.
ρ	=	time preference.
t	=	time.
F	=	utility function.
K	=	physical capital.
$T\ (A,K,I)$	=	PPF.
$U\ (A,K,I)$	$\equiv$	$u(T(A,K,I))$; $u(\cdot)$: utility function.
δ	=	depreciation rate.
φ	=	weight given to investment as to the formation of human capital.
γ_i, i	=	0,1,2: co-state variables.
$H(\cdot)$	=	Hamiltonian.
$*$	=	denotes steady states.
J	=	Jacobian.
a_{ij}	=	element of the ith row and jth column of a matrix.
W_i, i	=	1,2: constant.
λ	=	eigenvalues.
a, a_1, a_2, a_3, b, b_1, b_2, c	=	constants.
V_i, $i = 1,2$	=	measure for the degree of complementarity of physical and human capital.
β_2	=	coefficient giving the stability of the limit cycle.